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OPTIMAL ELASTIC STRUCTURES WITH FREQUENCY-DEPENDENT ELASTIC SUPPORTS

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Abstract—The optimal solution to the optimum design problem of elastic frames with frequency-dependent supports for specified fundamental natural frequency is shown to coincide with that of the frames with the corresponding frequency-independent supports, provided that the support stiffnesses of the former are expressed as single-valued non-increasing positive functions of frequency. Two new theorems are introduced for establishing one-to-one correspondence between the design spaces of these two classes of frames. It is also shown that the former optimal solution for a frame with such support characteristics is also the solution to the optimum design problem subject to the corresponding inequality constraint on fundamental natural frequency.

1. INTRODUCTION

The purpose of this paper is to disclose the characteristics of an optimal elastic structure consisting of members or elements with frequency-independent stiffnesses and supported by elastic springs or members with prescribed frequency-dependent stiffnesses. Building structures and other civil engineering structures are supported by the ground and a better model representing the restoring-force characteristics of the supporting ground is a set of springs or members with frequency-dependent stiffnesses.

The problem of optimum design of elastic structures for specified fundamental natural frequency has been investigated extensively. A number of theories and various numerical methods have been presented so far (Pierson, 1972; Olhoff, 1980; Haug and Cea, 1981). To the best of the authors' knowledge, however, no previous paper has dealt with a problem of optimum design of an elastic structure supported by members with prescribed frequency-dependent stiffnesses.

The eigenvalue problem of free vibration of such a structure has a mathematical structure different from that of a structure involving no spring or member with frequency-dependent stiffness. The former turns out to have a distinct stiffness matrix for each vibration mode and the orthogonality relation between any pair of eigenvectors will not hold in general. Furthermore, Rayleigh's principle will no longer hold without certain restriction for the former elastic structure.

In this paper, some characteristics of an elastic structure with a prescribed fundamental natural frequency supported by frequency-dependent springs are illustrated first by a simple example. Two new theorems are then introduced and proved for establishing one-to-one correspondence between the design spaces of an ordered set of elastic frames supported by members with frequency-dependent stiffnesses and of the corresponding ordered set of elastic frames supported by those with the corresponding frequency-independent stiffnesses, both with respect to fundamental natural frequency. It is shown also that the optimal solution to the problem of optimum design of the former structure for specified fundamental natural frequency is also the optimal solution to the problem subject to the corresponding inequality constraint on fundamental natural frequency, provided that the stiffnesses of the supporting members are single-valued non-increasing positive functions of frequency. An optimum design of a plane shear building model supported by elastic springs with frequency-dependent stiffnesses is illustrated for demonstrating the implication of the theorems.

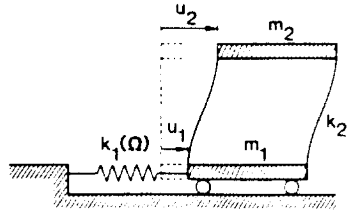

Fig. 1. Two-degree-of-freedom model.

2. ELASTIC STRUCTURES WITH FREQUENCY-DEPENDENT ELASTIC SUPPORT AND WITH FREQUENCY-INDEPENDENT ELASTIC SUPPORT

Some characteristics of an elastic structure with frequency-dependent elastic supports can be illustrated by the simple shear building model shown in Fig. 1. The model has two degrees of freedom only in horizontal direction. The stiffness of the support is to be frequency-dependent and is denoted by $k_1(\Omega)$ where Ω denotes the square of a natural circular frequency ω. On the other hand, the stiffness k_2 of the first story is to be frequency-independent. Let m_1 and m_2 denote the lumped masses of the ground and second floors, respectively. Then the corresponding eigenvalue problem is defined as follows.

$$\left[\Omega\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} - \begin{bmatrix} k_1(\Omega)+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\right]\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1}$$

where u_1 and u_2 denote the horizontal displacements of masses m_1 and m_2, respectively.

Let Ω_a denote the specified eigenvalue for the model. Summation of the first and second equations of eqn (1) under the condition that $\Omega = \Omega_a$ provides

$$\Omega_a(m_1u_1+m_2u_2) = k_1(\Omega_a)u_1 \tag{2}$$

from which the relative displacement ratio is given by

$$\frac{u_1}{u_2-u_1} = \frac{\Omega_a m_2}{k_1(\Omega_a)-\Omega_a(m_1+m_2)}. \tag{3}$$

The second equation of eqn (1) may be reduced to the following form.

$$k_2 = \Omega_a m_2\left[1+\frac{u_1}{u_2-u_1}\right]. \tag{4}$$

Substitution of eqn (3) into eqn (4) yields the stiffness k_2^* corresponding to Ω_a.

$$k_2^* = \frac{\Omega_a m_2\{k_1(\Omega_a)-\Omega_a m_1\}}{k_1(\Omega_a)-\Omega_a(m_1+m_2)}. \tag{5}$$

The eigenvalue equation for the elastic structure with $k_1(\Omega)$ and k_2^* may be written as follows.

$$f_{FD}(\Omega) = m_1m_2\Omega^2-[m_1k_2^*+m_2\{k_1(\Omega)+k_2^*\}]\Omega+k_1(\Omega)k_2^* = 0. \tag{6}$$

On the other hand, the eigenvalue equation for the elastic structure with the frequency-independent support $\bar{k}_1 = k_1(\Omega_a)$ and k_2^* may be expressed as follows.

$$f_{FI}(\Omega) = m_1m_2\Omega^2-[m_1k_2^*+m_2(\bar{k}_1+k_2^*)]\Omega+\bar{k}_1k_2^* = 0. \tag{7}$$

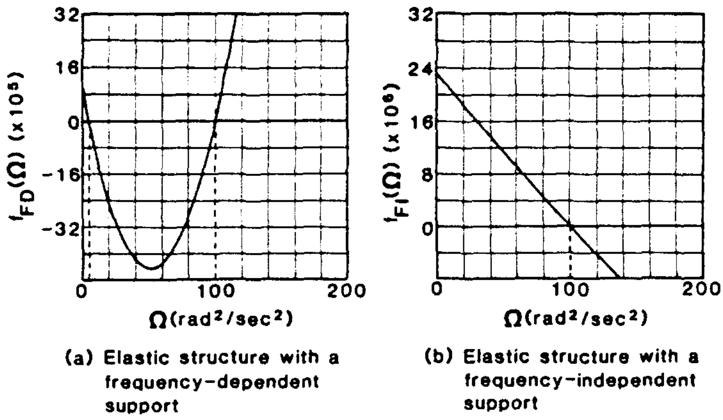

Fig. 2. Plots of $f_{FD}(\Omega)$ in eqn (6) and $f_{FI}(\Omega)$ in eqn (7) with respect to Ω in case of $k_1(\Omega) = 0.2\Omega^2 + 100$ ($\times 10^3$ kg$\cdot$rad^2/s^2).

Consider a model with $m_1 = m_2 = 1.0 \times 10^4$ (kg), $\Omega_a = 100.0$ (rad^2/s^2) and $k_1(\Omega) = 0.2\Omega^2 + 100(\times 10^3$ kg$\cdot$rad^2/s^2). Figures 2(a) and (b) show the plots of $f_{FD}(\Omega)$ in eqn (6) and $f_{FI}(\Omega)$ in eqn (7) for this model with respect to Ω, respectively. Figures 2(a) and (b) clearly illustrate that even if the elastic structure with the frequency-independent support $\bar{k}_1 = k_1(\Omega_a)$ has Ω_a as the fundamental eigenvalue, the elastic structure with the frequency-dependent support $k_1(\Omega)$ does not necessarily have Ω_a as the fundamental eigenvalue.

Consider next another model also with $m_1 = m_2 = 1.0 \times 10^4$ (kg), with the frequency-dependent support $k_1(\Omega) = \Omega - 20$ ($\times 10^3$ kg$\cdot$rad^2/s^2), and with $\Omega_a = 100.0$ (rad^2/s^2) as an eigenvalue. The stiffness k_2^* is determined similarly from eqn (2) through eqn (5). Then the eigenvalue equations for the elastic structures with $\{k_1(\Omega), k_2^*\}$ and with $\{\bar{k}_1 = k_1(\Omega_a), k_2^*\}$ are given by eqn (6) and eqn (7), respectively. Figures 3(a) and (b) show the plots of $f_{FD}(\Omega)$ and $f_{FI}(\Omega)$ with respect to Ω in this case, respectively. Figures 3(a) and (b) illustrate that even if the elastic structure with the frequency-dependent support $k_1(\Omega)$ has Ω_a as the fundamental eigenvalue, the elastic structure with the frequency-independent support $\bar{k}_1 = k_1(\Omega_a)$ does not necessarily have Ω_a as the fundamental eigenvalue.

3. OPTIMUM DESIGN PROBLEM FOR SPECIFIED FUNDAMENTAL NATURAL FREQUENCY

Consider an elastic framed structure, shown in Fig. 4, consisting of uniform elastic members with frequency-independent stiffnesses and supported by elastic members with prescribed frequency-dependent stiffnesses. The cross-sectional areas of the frequency-independent elastic members are chosen as design variables. The centerline dimensions of

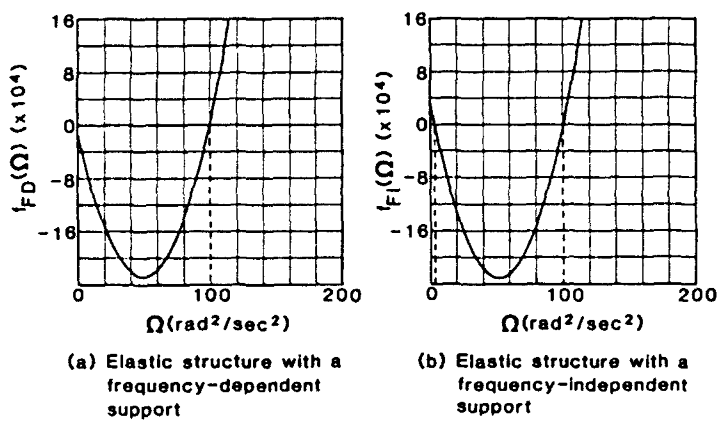

Fig. 3. Plots of $f_{FD}(\Omega)$ in eqn (6) and $f_{FI}(\Omega)$ in eqn (7) with respect to Ω in case of $k_1(\Omega) = \Omega - 20$ ($\times 10^3$ kg$\cdot$rad^2/s^2).

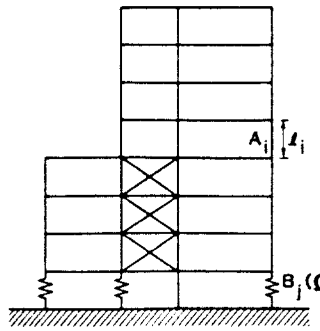

Fig. 4. Elastic frame with frequency-dependent elastic supports.

the frame and of the supporting members are prescribed. The degrees of freedom of the whole structure, the number of members of the frame and the number of independent stiffnesses of the supporting members are denoted by N, n and S, respectively. Let A_i and l_i denote the cross-sectional area and length, respectively, of the ith member of the frame. All the members that are to be assigned one and the same cross-sectional area are to be regarded as one member with the whole length. The set of the cross-sectional areas A_i ($i = 1, 2, \ldots, n$) is represented by $\mathbf{A}$ and is called "design $\mathbf{A}$". The frame may be composed of some truss members with axial stiffness only and/or of some beams or columns with axial and bending stiffnesses. The Young's modulus of each member is to be prescribed. The moment of inertia of the cross section of each member with bending stiffness may be a nonlinear function of the cross-sectional area.

The supporting members may also be composed of some truss members and/or some beams or columns whose axial and/or bending stiffnesses are frequency-dependent. The jth axial or bending stiffness of the supporting members is denoted by $B_j(\Omega)$ and a set of $B_j(\Omega)$'s is represented by $\mathbf{B}(\Omega) = \{B_j(\Omega)\}$. For the sake of simplicity, the elastic frames of design $\mathbf{A}$ supported by the members with frequency-dependent stiffnesses $\mathbf{B}(\Omega)$ and by the members with frequency-independent stiffnesses $\bar{\mathbf{B}}_a$ ($= \mathbf{B}(\Omega_a)$) are referred to in the following as "the frame of design $\mathbf{A}$ with $\mathbf{B}(\Omega)$" and "the frame of design $\mathbf{A}$ with $\bar{\mathbf{B}}_a$", respectively.

The stiffness matrix $\mathbf{K}_{FD}(\mathbf{A}; \mathbf{B}(\Omega))$ of $N \times N$ in the global coordinate system of the frame of design $\mathbf{A}$ with $\mathbf{B}(\Omega)$ may be expressed as follows in terms of the stiffness matrix $\mathbf{K}_I(\mathbf{A})$ associated with the frame and the stiffness matrix $\mathbf{K}_{II}(\mathbf{B}(\Omega))$ associated with the supporting members.

$$\mathbf{K}_{FD}(\mathbf{A}; \mathbf{B}(\Omega)) = \mathbf{K}_I(\mathbf{A}) + \mathbf{K}_{II}(\mathbf{B}(\Omega)). \tag{8}$$

Both $\mathbf{K}_I(\mathbf{A})$ and $\mathbf{K}_{II}(\mathbf{B}(\Omega))$ are the stiffness matrices of $N \times N$ in the global coordinate system. Each component of $\mathbf{K}_{II}(\mathbf{B}(\Omega))$ is a linear function of some of $B_j(\Omega)$'s.

The mass matrix of the frames of design $\mathbf{A}$ with $\mathbf{B}(\Omega)$ and with $\bar{\mathbf{B}}_a$ is to consist of the lumped mass matrix $\mathbf{M}_n$ of $N \times N$ and the consistent mass matrix $\mathbf{M}_s(\mathbf{A})$ of $N \times N$. The mass matrix $\mathbf{M}(\mathbf{A})$ may then be expressed as follows.

$$\mathbf{M}(\mathbf{A}) = \mathbf{M}_s(\mathbf{A}) + \mathbf{M}_n. \tag{9}$$

If the kth-order eigenvalue and the kth-order eigenvector of the frame of design $\mathbf{A}$ with $\mathbf{B}(\Omega)$ are denoted by Ω_k and $\mathbf{Z}_{FD}^{(k)}$, respectively, then the eigenvalue problem in this case may be expressed as follows.

$$[\mathbf{K}_{FD}(\mathbf{A}; \mathbf{B}(\Omega_k)) - \Omega_k \mathbf{M}(\mathbf{A})]\mathbf{Z}_{FD}^{(k)} = \mathbf{0}. \tag{10}$$

Equation (10) indicates that this eigenvalue problem has a mathematical structure different from that of a usual elastic structure without any member with frequency-dependent stiffness.

The minimum weight design problem of this model structure for specified fundamental natural frequency may be stated as follows.

Problem FEC

For an elastic frame supported by elastic members with frequency-dependent stiffnesses $\mathbf{B}(\Omega)$, find $\mathbf{A}$ that minimizes the objective function

$$w = \mathbf{A}^T\mathbf{L} \tag{11}$$

subject to the constraint on fundamental natural frequency

$$\omega_1(\mathbf{A}) = \omega_a \quad (\text{or } \Omega_1(\mathbf{A}) = \Omega_a) \tag{12}$$

and the constraints on the minimum cross-sectional areas

$$A_i \geq \bar{A}_i \quad (i = 1, 2, \ldots, n) \tag{13}$$

where $\omega_1(\mathbf{A})$ and ω_a denote the fundamental natural frequency of the frame of design $\mathbf{A}$ with $\mathbf{B}(\Omega)$ and the specified fundamental natural frequency, respectively. Furthermore, $\Omega_1(\mathbf{A}) = \omega_1(\mathbf{A})^2$, $\Omega_a = \omega_a^2$ and $\mathbf{L}^T = \{l_1 \ldots l_n\}$ where $(\ )^T$ indicates the transpose of a vector. The case where all of the constraints on the minimum cross-sectional areas are satisfied in equality is not dealt with here.

In this paper, the only case will be considered where all of the stiffnesses $\mathbf{B}(\Omega)$ of the supporting members are expressed as single-valued non-increasing positive functions of Ω. It may appear difficult in *Problem FEC* to distinguish the frame of design $\mathbf{A}$ with $\mathbf{B}(\Omega)$ from the frame of design $\mathbf{A}$ with $\bar{\mathbf{B}}_a$ after the specification of the fundamental natural frequency. It should be noted, however, that while the fundamental eigenvector of the frame of design $\mathbf{A}$ with $\bar{\mathbf{B}}_a$ is characterized by Rayleigh's principle, that of the frame of design $\mathbf{A}$ with $\mathbf{B}(\Omega)$ is not. Even if an eigenvector might be found for the frame of design $\mathbf{A}$ with $\mathbf{B}(\Omega)$, no other method to confirm it as the fundamental eigenvector exists except by demonstrating that the corresponding eigenvalue is indeed the minimum positive root of the eigenvalue equation. As demonstrated in the previous section, there exist distinct differences between a frame with supports $\mathbf{B}(\Omega)$ and a frame with supports $\bar{\mathbf{B}}_a$ in the case that the stiffnesses $\mathbf{B}(\Omega)$ have no restriction on their characteristics.

4. TWO THEOREMS FOR ELASTIC FRAMES WITH FREQUENCY-DEPENDENT AND FREQUENCY-INDEPENDENT SUPPORTS

In order to derive a set of necessary and sufficient conditions for global optimality to *Problem FEC*, the following theorems must be introduced and proved first.

Theorem 1. *Let ω_a denote the fundamental natural frequency of the frame of design* $\mathbf{A}$ *with* $\bar{\mathbf{B}}_a$ $(=\mathbf{B}(\Omega_a))$. *Then the frame of design* $\mathbf{A}$ *with* $\mathbf{B}(\Omega)$, *all the elements $B_j(\Omega)$ of which are single-valued non-increasing positive functions of Ω, has the same set of the fundamental natural frequency and the fundamental eigenvector as that of the former frame.*

Proof. Theorem 1 may be proved by showing that the frame of design $\mathbf{A}$ with $\mathbf{B}(\Omega)$ has ω_a as one of the natural frequencies but will not have any other natural frequencies smaller than ω_a.

Equation (10) for the frame of design $\mathbf{A}$ with $\bar{\mathbf{B}}_a$ and with the fundamental natural frequency ω_a may be written as follows.

$$[\mathbf{K}_{Fl}(\mathbf{A}\,;\bar{\mathbf{B}}_a) - \Omega_a\mathbf{M}(\mathbf{A})]\mathbf{Z}_{Fl}(\mathbf{A}) = \mathbf{0} \tag{14}$$

where $\mathbf{K}_{Fl}(\mathbf{A}\,;\bar{\mathbf{B}}_a)$ and $\mathbf{Z}_{Fl}(\mathbf{A})$ denote the stiffness matrix of the frame of design $\mathbf{A}$ with $\bar{\mathbf{B}}_a$ and the fundamental eigenvector of this frame, respectively. $\mathbf{K}_{Fl}(\mathbf{A}\,;\bar{\mathbf{B}}_a)$ is the matrix derived by replacing $\mathbf{B}(\Omega)$ by $\bar{\mathbf{B}}_a$ in the matrix $\mathbf{K}_{FD}(\mathbf{A}:\mathbf{B}(\Omega))$. Premultiplication of eqn (14) by $\mathbf{Z}_{FI}(\mathbf{A})^T$ provides the following expression of Ω_a as a Rayleigh's quotient.

$$\Omega_a = \frac{\mathbf{Z}_{FI}(\mathbf{A})^T\mathbf{K}_{FI}(\mathbf{A};\bar{\mathbf{B}}_a)\mathbf{Z}_{FI}(\mathbf{A})}{\mathbf{Z}_{FI}(\mathbf{A})^T\mathbf{M}(\mathbf{A})\mathbf{Z}_{FI}(\mathbf{A})}. \tag{15}$$

Since eqn (15) is the Rayleigh's quotient for the frame consisting of members with frequency-independent stiffnesses and Ω_a is indeed the fundamental eigenvalue of this frame, the right-hand side of eqn (15) for any other kinematically admissible modes has the minimum value Ω_a due to Rayleigh's principle.

Now assume that the frame of design $\mathbf{A}$ with $\mathbf{B}(\Omega)$ has an eigenvalue $\Omega_p = \Omega_a - \Delta\Omega(\Delta\Omega > 0)$ smaller than Ω_a. Then the stiffness matrix of this frame for the free vibration of frequency ω_p may be expressed as follows due to the linearity of each component of the matrix $\mathbf{K}_{II}$ with respect to $B_j(\Omega)$.

$$\mathbf{K}_{FD}(\mathbf{A};\mathbf{B}(\Omega_a - \Delta\Omega)) = \mathbf{K}_I(\mathbf{A}) + \mathbf{K}_{II}(\mathbf{B}(\Omega_a)) + \mathbf{K}_{II}(\Delta\mathbf{B}) \tag{16}$$

where $\Delta\mathbf{B} = \mathbf{B}(\Omega_a - \Delta\Omega) - \mathbf{B}(\Omega_a)$ and $\Delta B_j \geqslant 0$ (for all j) due to the non-increasing characteristics of $\mathbf{B}(\Omega)$. Let $\mathbf{Z}_p(\mathbf{A})$ denote the eigenvector of this frame corresponding to the eigenvalue Ω_p. By substituting Ω_p and $\mathbf{Z}_p(\mathbf{A})$ in place of Ω_k and $\mathbf{Z}_{FD}^{(k)}$ in eqn (10) and premultiplying the resulting equation by $\mathbf{Z}_p(\mathbf{A})^T$, the eigenvalue Ω_p may be written as follows.

$$\Omega_p = \frac{\mathbf{Z}_p(\mathbf{A})^T\mathbf{K}_{FD}(\mathbf{A};\mathbf{B}(\Omega_a - \Delta\Omega))\mathbf{Z}_p(\mathbf{A})}{\mathbf{Z}_p(\mathbf{A})^T\mathbf{M}(\mathbf{A})\mathbf{Z}_p(\mathbf{A})}. \tag{17}$$

Substitution of eqn (16) into eqn (17) with the aid of $\mathbf{K}_{II}(\mathbf{B}(\Omega_a)) = \mathbf{K}_{II}(\bar{\mathbf{B}}_a)$ yields the following equation.

$$\Omega_p = \frac{\mathbf{Z}_p(\mathbf{A})^T\mathbf{K}_{FI}(\mathbf{A};\bar{\mathbf{B}}_a)\mathbf{Z}_p(\mathbf{A})}{\mathbf{Z}_p(\mathbf{A})^T\mathbf{M}(\mathbf{A})\mathbf{Z}_p(\mathbf{A})} + \frac{\mathbf{Z}_p(\mathbf{A})^T\mathbf{K}_{II}(\Delta\mathbf{B})\mathbf{Z}_p(\mathbf{A})}{\mathbf{Z}_p(\mathbf{A})^T\mathbf{M}(\mathbf{A})\mathbf{Z}_p(\mathbf{A})}. \tag{18}$$

Since $\mathbf{Z}_p(\mathbf{A})$ is a kinematically admissible mode for the frame of design $\mathbf{A}$ with $\bar{\mathbf{B}}_a$, the following inequality is drawn from eqn (15) due to Rayleigh's principle.

$$\Omega_a \leqslant \frac{\mathbf{Z}_p(\mathbf{A})^T\mathbf{K}_{FI}(\mathbf{A};\bar{\mathbf{B}}_a)\mathbf{Z}_p(\mathbf{A})}{\mathbf{Z}_p(\mathbf{A})^T\mathbf{M}(\mathbf{A})\mathbf{Z}_p(\mathbf{A})}. \tag{19}$$

Furthermore, the fact that $\Delta B_j \geqslant 0$ (for all j) and the positive definiteness of the matrices $\mathbf{K}_{Bj}$ and $\mathbf{M}(\mathbf{A})$ provide the following inequalities.

$$\mathbf{Z}_p(\mathbf{A})^T\mathbf{K}_{II}(\Delta\mathbf{B})\mathbf{Z}_p(\mathbf{A}) = \sum_{j=1}^{S} \Delta B_j\{\mathbf{Z}_p(\mathbf{A})^T\mathbf{K}_{Bj}\mathbf{Z}_p(\mathbf{A})\} \geqslant 0 \tag{20a}$$

$$\mathbf{Z}_p(\mathbf{A})^T\mathbf{M}(\mathbf{A})\mathbf{Z}_p(\mathbf{A}) > 0 \tag{20b}$$

where $\mathbf{K}_{Bj}$ is defined as follows.

$$\mathbf{K}_{II}(\Delta\mathbf{B}) = \sum_{j=1}^{S} \Delta B_j \mathbf{K}_{Bj}. \tag{21}$$

In inequalities (20a, b), $\mathbf{Z}_p(\mathbf{A})^T\mathbf{M}(\mathbf{A})\mathbf{Z}_p(\mathbf{A})$ and $\mathbf{Z}_p(\mathbf{A})^T\mathbf{K}_{Bj}\mathbf{Z}_p(\mathbf{A})$ represent the total kinetic energy of this frame in the free vibration and the strain energy per unit stiffness of the jth element with $B_j(\Omega)$, respectively. Equation (18) and inequalities (19) and (20a,b) require that the inequality $\Omega_p \geqslant \Omega_a$ must hold. But this result apparently contradicts the initial assumption that $\Omega_p < \Omega_a$. Therefore there does not exist any eigenvalue smaller than Ω_a in the case that all of the stiffnesses $\mathbf{B}(\Omega)$ of the supporting members are single-valued non-

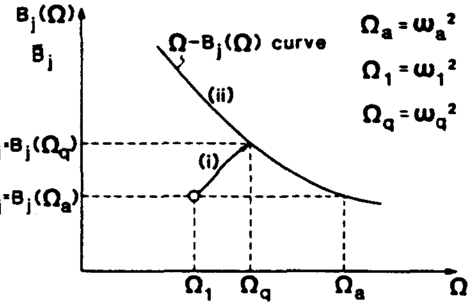

Fig. 5. (i) Plots of fundamental eigenvalues of the frame of design A with respect to stiffness $\bar{B}_j$ of a frequency-independent supporting member and (ii) $\Omega - B_j(\Omega)$ curve.

increasing positive functions of Ω. Hence it is concluded that the eigenvalue Ω_a of this frame of design A with $\mathbf{B}(\Omega)$ is indeed the fundamental eigenvalue.

Since the frame of design A with $\mathbf{B}(\Omega)$ and the frame of design A with $\bar{\mathbf{B}}_a$ have the same stiffness and mass matrices due to $\mathbf{K}_{FD}(\mathbf{A}\,;\,\mathbf{B}(\Omega_a)) = \mathbf{K}_{FI}(\mathbf{A}\,;\,\bar{\mathbf{B}}_a)$, it is evident that they have the same fundamental eigenvector. This completes the proof.

The converse theorem may be stated as follows.

Theorem 2. Let ω_a *denote the fundamental natural frequency of the frame of design* A *with* $\mathbf{B}(\Omega)$ *all the elements* $B_j(\Omega)$ *of which are single-valued non-increasing positive functions of* Ω. *Then the frame of design* A *with* $\bar{\mathbf{B}}_a$ $(=\mathbf{B}(\Omega_a))$ *has the same set of the fundamental natural frequency and the fundamental eigenvector as that of the former frame.*

Proof. Theorem 2 can be proved by showing that the frame of design A with $\bar{\mathbf{B}}_a$ has ω_a as one of the natural frequencies, but will not have any other natural frequencies smaller than ω_a.

Assume that the frame of design A with $\bar{\mathbf{B}}_a = \mathbf{B}(\Omega_a) = \{\bar{B}_{aj}\}$ has ω_a as one of the natural frequencies other than the fundamental natural frequency. Then this frame apparently has the fundamental natural frequency ω_1 smaller than ω_a. It is possible to prove that the frame of design A has a fundamental natural frequency greater than or equal to ω_1 when supported by elastic members with stiffnesses larger than $\bar{\mathbf{B}}_a$. Figure 5 indicates that the plots of the fundamental eigenvalue of the frame of design A with respect to the stiffness of a frequency-independent supporting member intersect with the $\Omega - B_j(\Omega)$ curve at a point smaller than Ω_a. Let Ω_q and $\bar{\mathbf{B}}_q = \{\bar{B}_{qj}\}$ denote the fundamental eigenvalue and the stiffnesses of the frequency-independent supporting members at the intersection. Then it is apparent that $\bar{\mathbf{B}}_q = \mathbf{B}(\Omega_q)$. It follows from Theorem 1 that if the frame of design A with $\bar{\mathbf{B}}_q = \mathbf{B}(\Omega_q)$ has the fundamental natural frequency ω_q, the frame of design A with $\mathbf{B}(\Omega)$ also has the fundamental natural frequency ω_q. This consequence, $\omega_q < \omega_a$, apparently contradicts the condition that the frame of design A with $\mathbf{B}(\Omega)$ has the fundamental natural frequency ω_a. It is therefore concluded that the frame of design A with $\bar{\mathbf{B}}_a$ has ω_a as the fundamental natural frequency. This completes the proof.

Theorems 1 and 2 lead us to the conclusion that the design space with respect to the fundamental natural frequency ω_a of a frame of design A with $\mathbf{B}(\Omega)$ has one-to-one correspondence with that of the frame of design A with $\bar{\mathbf{B}}_a = \mathbf{B}(\Omega_a)$ so long as all the elements of $\mathbf{B}(\Omega)$ are single-valued non-increasing positive functions of Ω.

The necessary and sufficient conditions for global optimality to *Problem FEC* can be derived as follows after Sheu (1968) for the case where the moment of inertia of the cross section of each member is expressed as a linear function of the cross-sectional area.

$$\frac{\mathbf{Z}_{FD}(\tilde{\mathbf{A}})^T(\mathbf{K}_i - \Omega_a \mathbf{M}_i)\mathbf{Z}_{FD}(\tilde{\mathbf{A}})}{\mathbf{Z}_{FD}(\tilde{\mathbf{A}})^T\mathbf{M}(\tilde{\mathbf{A}})\mathbf{Z}_{FD}(\tilde{\mathbf{A}})} = \frac{l_i}{\mu} \quad \text{if } \tilde{A}_i > \bar{A}_i \tag{22a}$$

$$\frac{\mathbf{Z}_{FD}(\tilde{\mathbf{A}})^T(\mathbf{K}_i - \Omega_a \mathbf{M}_i)\mathbf{Z}_{FD}(\tilde{\mathbf{A}})}{\mathbf{Z}_{FD}(\tilde{\mathbf{A}})^T\mathbf{M}(\tilde{\mathbf{A}})\mathbf{Z}_{FD}(\tilde{\mathbf{A}})} \leqslant \frac{l_i}{\mu} \quad \text{if } \tilde{A}_i = \bar{A}_i \tag{22b}$$

where $\tilde{\mathbf{A}}$ is the optimum design for *Problem FEC*, μ a positive scalar and $\mathbf{K}_i$, $\mathbf{M}_i$ the following matrices.

$$\mathbf{K}_i = \frac{\partial \mathbf{K}_f(\mathbf{A})}{\partial A_i}, \quad \mathbf{M}_i = \frac{\partial \mathbf{M}_f(\mathbf{A})}{\partial A_i}. \tag{23a,b}$$

5. OPTIMUM DESIGN PROBLEM SUBJECT TO AN INEQUALITY CONSTRAINT ON FUNDAMENTAL NATURAL FREQUENCY

Thus far, only the optimum design problem of elastic structures supported by members with frequency-dependent stiffnesses subject to an equality constraint on fundamental natural frequency has been dealt with. Consider next an extended optimum design problem subject to an inequality constraint on fundamental natural frequency. For the sake of simplicity, an optimum design problem without any constraint on minimum cross-sectional area is considered. The optimum design problem of elastic structures supported by members with frequency-dependent stiffnesses subject only to an inequality constraint on fundamental natural frequency may be stated as follows.

Problem FICA

For an elastic frame supported by elastic members with frequency-dependent stiffnesses $\mathbf{B}(\Omega)$, find $\mathbf{A}$ that minimizes the objective function (11) subject to the inequality constraint on fundamental natural frequency

$$\omega_1(\mathbf{A}) \geqslant \omega_a \quad (\text{or } \Omega_1(\mathbf{A}) \geqslant \Omega_a). \tag{24}$$

A problem where all the constraints on the minimum cross-sectional areas are deleted in *Problem FEC* is called *Problem FECA* in the following.

Since the stiffnesses $\mathbf{B}(\Omega)$ of the supporting members are frequency-dependent, the optimal solution to *Problem FICA* may not necessarily coincide with the optimal solution to the corresponding *Problem FECA*. In this section, the qualification condition on $\mathbf{B}(\Omega)$ is discussed so that the optimal solution to *Problem FECA* is also the optimal solution to *Problem FICA*. The optimality conditions characterizing the global optimality of the solution to *Problem FECA* have been stated in the previous section. A set of optimal solutions to *Problem FECA* for a specified range of fundamental eigenvalue Ω constitute an ordered set of optimum designs (Nakamura and Ohsaki, 1988). All the variables in this ordered set of optimum designs may be regarded as piecewisely differentiable continuous functions of Ω. If the following condition is satisfied for all Ω in the range of $\Omega \geqslant \Omega_a$, then the solution to *Problem FECA* with the constraint $\Omega = \Omega_a$ also becomes the solution to *Problem FICA* with the constraint $\Omega \geqslant \Omega_a$.

$$\frac{\mathrm{d}w(\Omega)}{\mathrm{d}\Omega} > 0. \tag{25}$$

While the concepts of an ordered set of optimum designs and of regarding the optimum design variables as functions of some problem parameters are known (see for instance, Nakamura and Nagase, 1976), some explicit general expressions of sensitivity coefficients of optimum solutions have been introduced rather recently (Sobieszczanski-Sobieski *et al.*, 1982; Schmit and Chang, 1984; Vanderplaats and Yoshida, 1985). The derivative of $w(\Omega)$ can be written explicitly as follows.

Let $\mathbf{A}(\Omega)$ and $\hat{\mathbf{Z}}_{FD}(\Omega) \equiv \mathbf{Z}_{FD}(\mathbf{A}(\Omega))$ denote the cross-sectional areas and the fundamental eigenvector of the optimal frame supported by members with frequency-dependent stiffnesses in *Problem FECA* with the constraint on fundamental eigenvalue $\Omega_1(\mathbf{A}) = \Omega$. Since the constraint on fundamental natural frequency is the only active constraint in *Problem FECA*, the derivative of the objective function with respect to Ω can be expressed as follows after Barthelemy and Sobieszczanski-Sobieski (1983).

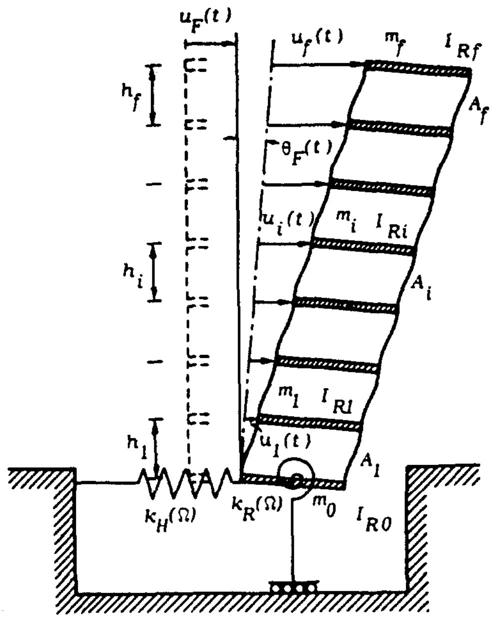

Fig. 6. Plane elastic shear building supported by two frequency-dependent elastic springs.

$$\frac{\mathrm{d}w(\Omega)}{\mathrm{d}\Omega} = \mu(\Omega)\left[1 - \sum_{j=1}^{S} \frac{\mathrm{d}B_j(\Omega)}{\mathrm{d}\Omega}\,\frac{\hat{\mathbf{Z}}_{FD}(\Omega)^T\mathbf{K}_{Bj}\hat{\mathbf{Z}}_{FD}(\Omega)}{\hat{\mathbf{Z}}_{FD}(\Omega)^T\hat{\mathbf{M}}(\Omega)\hat{\mathbf{Z}}_{FD}(\Omega)}\right] \tag{26}$$

where $\hat{\mathbf{M}}(\Omega) \equiv \mathbf{M}(\mathbf{A}(\Omega))$ and $\mu(\Omega)$ denotes a positive Lagrange multiplier. As stated before, $\hat{\mathbf{Z}}_{FD}(\Omega)^T\hat{\mathbf{M}}(\Omega)\hat{\mathbf{Z}}_{FD}(\Omega)$ and $\hat{\mathbf{Z}}_{FD}(\Omega)^T\mathbf{K}_{Bj}\hat{\mathbf{Z}}_{FD}(\Omega)$ in eqn (26) are positive definite.

If all the elements of $\mathbf{B}(\Omega)$ are single-valued non-increasing positive functions of Ω, then the following inequalities hold.

$$\frac{\mathrm{d}B_j(\Omega)}{\mathrm{d}\Omega} \leqslant 0 \quad (\text{for all } j). \tag{27}$$

It can be shown that if inequalities (27) are utilized in eqn (26), then inequality (25) always holds. It is concluded therefore that the solution to *Problem FECA* is also the solution to *Problem FICA*, provided inequalities (27) are satisfied, i.e. all the stiffnesses of the supporting members are given as single-valued non-increasing positive functions of Ω.

It should be noted that $\mathrm{d}w(\Omega)/\mathrm{d}\Omega$ and $\mathrm{d}B_j(\Omega)/\mathrm{d}\Omega$ in eqn (26) need only to be piecewisely differentiable, e.g. the stiffness functions $\mathbf{B}(\Omega)$ of the supporting members may include jumps.

6. EXAMPLE: PLANE ELASTIC SHEAR BUILDING SUPPORTED BY FREQUENCY-DEPENDENT ELASTIC SPRINGS

Consider a plane elastic shear building, shown in Fig. 6, supported by two elastic springs with frequency-dependent stiffnesses as an example. The stiffnesses $k_H(\Omega)$ and $k_R(\Omega)$ of the two springs are to be prescribed. The s columns with equal stiffness in each story are connected by rigid floors. The lumped mass in the ith floor, its moment of inertia around its centroid and the story height of the ith story are denoted by m_i, I_{Ri} and h_i, respectively, and are prescribed. Let r_i and E denote the radius of gyration of the cross section of the columns in the ith story and Young's modulus of all the columns and they are also prescribed. The set of cross-sectional areas $\mathbf{A}$ of the columns are the design variables here. For simplicity of expression, the consistent mass matrix is not considered in this example. A horizontal displacement $u_F(t)$ and an angle $\theta_F(t)$ of rotation of the base and a set of relative horizontal displacements $u_1(t)\ldots u_f(t)$ of the shear building to the base are chosen as generalized coordinates, where t denotes time. The fundamental eigenvector of the shear building of design $\mathbf{A}$ with $k_H(\Omega)$ and $k_R(\Omega)$ is expressed here as $\mathbf{Z}_{FD}(\mathbf{A})^T = \{U_F(\mathbf{A})\ \ U_F(\mathbf{A})+U_1(\mathbf{A})\ldots U_F(\mathbf{A})+U_f(\mathbf{A})\ \ \Theta_F(\mathbf{A})\}$. The degrees of freedom of this model is $N = f+2$.

The matrices $\mathbf{M}_n$, $\mathbf{K}_i$ and $\mathbf{K}_{Bj}$ defined in the foregoing sections may be expressed as follows.

$$\mathbf{M}_n = \begin{bmatrix} m_0 & & & & 0 \\ & & 0 & & \\ & m_1 & & & m_1H_1 \\ 0 & & \ddots & & \vdots \\ & & & m_f & m_fH_f \\ 0 & m_1H_1 & \ldots & m_fH_f & I_T \end{bmatrix}, \quad \mathbf{K}_i = \begin{bmatrix} 0 & & 0 & & & 0 \\ & \ddots\,0 & & \ddots\,0 & & \\ 0 & & & & & \\ & \ddots\,0 & g_i & -g_i & & \\ & & & & 0 & \\ & & -g_i & g_i & & \ddots\,0 \\ 0 & & & 0 & & 0 \\ & & & \ddots\,0 & & \ddots\,0 \end{bmatrix}$$

$$\mathbf{K}_{B1} = \begin{bmatrix} 1 & & & \\ & 0 & & 0 \\ & & \ddots & \\ & 0 & & 0 \\ & & & & 0 \end{bmatrix}, \quad \mathbf{K}_{B2} = \begin{bmatrix} 0 & & & \\ & 0 & & 0 \\ & & \ddots & \\ & 0 & & 0 \\ & & & & 1 \end{bmatrix} \qquad \text{(28a–d)}$$

where

$$H_i = \sum_{j=1}^{i} h_j, \quad I_T = \sum_{j=1}^{f} m_j H_j^2 + \sum_{j=0}^{f} I_{Rj}, \quad g_i = \frac{12sEr_i^2}{h_i^3}. \qquad \text{(29a–c)}$$

In this example, $B_1(\Omega) = k_H(\Omega)$ and $B_2(\Omega) = k_R(\Omega)$. The matrix $\mathbf{M}_i$ becomes the null matrix according to the assumption mentioned above.

The problem corresponding to *Problem FECA* may now be stated as follows.

Problem SEC

For a plane elastic shear building supported by two springs with frequency-dependent stiffnesses $k_H(\Omega)$ and $k_R(\Omega)$, find $\mathbf{A}$ that minimizes the objective function

$$w = \mathbf{A}^T\mathbf{L} \qquad (30)$$

subject to the constraint on fundamental natural frequency

$$\omega_1(\mathbf{A}) = \omega_a \qquad (31)$$

where $\mathbf{A}^T = \{A_1 \ldots A_f\}$ and $\mathbf{L}^T = \{h_1 \ldots h_f\}$.

Application of the optimality condition (22a) into *Problem SEC* provides the following equation.

$$\frac{\mathbf{Z}_{FD}(\tilde{\mathbf{A}})^T\mathbf{K}_i\mathbf{Z}_{FD}(\tilde{\mathbf{A}})}{\mathbf{Z}_{FD}(\tilde{\mathbf{A}})^T\mathbf{M}_n\mathbf{Z}_{FD}(\tilde{\mathbf{A}})} = \frac{h_i}{\mu} \quad (i = 1, 2, \ldots, f). \qquad (32)$$

Equation (32) may be rewritten as follows.

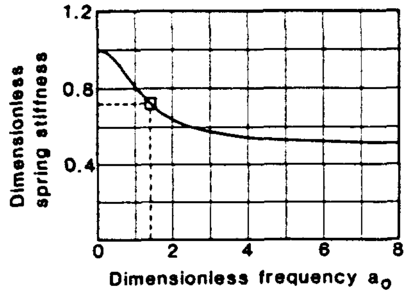

Fig. 7. Plots of dimensionless stiffness of a rotational spring with respect to dimensionless frequency.

$$\frac{g_i\,[U_i(\tilde{\Lambda})-U_{i-1}(\tilde{\Lambda})]^2}{\mathbf{Z}_{FD}(\tilde{\Lambda})^T\mathbf{M}_n\mathbf{Z}_{FD}(\tilde{\Lambda})}=\frac{h_i}{\mu}\quad(i=1,2,\ldots,f). \tag{33}$$

It should be noted that the optimality conditions (33) involve a problem of selecting an appropriate combination of square root signs. The set of optimality conditions (33) must therefore be understood as the conditions stated for the true fundamental eigenvector of the shear building of design $\tilde{\Lambda}$ with $\{k_H(\Omega), k_R(\Omega)\}$. Since it can be stated from Theorems 1 and 2 that the shear building of design $\tilde{\Lambda}$ with $\{k_H(\Omega), k_R(\Omega)\}$ and the shear building of design $\tilde{\Lambda}$ with $\{\bar{k}_H = k_H(\Omega_a), \bar{k}_R = k_R(\Omega_a)\}$ have the same set of the fundamental natural frequency and the fundamental eigenvector, this problem can be resolved by requiring that the eigenvector in eqn (33) is the true fundamental eigenvector of the shear building of design $\tilde{\Lambda}$ with $\{\bar{k}_H, \bar{k}_R\}$. It can be proved that if a set of all the positive roots or a set of all the negative roots is adopted, then the corresponding eigenvector indeed minimizes the Rayleigh's quotient for the shear building of design $\tilde{\Lambda}$ with $\{\bar{k}_H, \bar{k}_R\}$ and, therefore, is the true fundamental eigenvector. This circumstance is almost the same as in the case of a plane elastic shear building with a fixed base (Nakamura and Yamane, 1986).

It is possible to derive a set of closed-form solutions of the fundamental eigenvector and the optimal cross-sectional areas to *Problem SEC* by utilizing the optimality conditions (33) and the equations corresponding to eqn (10) for this model as in the case of elastic shear buildings supported by springs with frequency-independent stiffnesses (Nakamura and Takewaki, 1985).

Now in order to demonstrate the validity of Theorems 1 and 2, consider a rigid disc rested on an elastic half space and evaluate the frequency-dependent stiffnesses of the two elastic springs as the real parts of the impedance functions. Approximate analytical expressions for $k_H(\Omega)$ and $k_R(\Omega)$ have been derived as follows by Veletsos and Verbic (1974).

$$k_H(\Omega)=\frac{8\rho V_s^2 r_o}{2-v} \tag{34}$$

$$k_R(\Omega)=\frac{8\rho V_s^2 r_o^3}{3(1-v)}\left[1-b_1+\frac{b_1}{1+b_2^2\left(\frac{r_o}{V_s}\right)^2\Omega}-b_3\left(\frac{r_o}{V_s}\right)^2\Omega\right] \tag{35}$$

where V_s, ρ, v and r_o denote the shear wave velocity, the mass density and the Poisson's ratio of the half-space soil and the radius of the rigid disc, respectively. The coefficients b_1, b_2 and b_3 are the constants given corresponding to the Poisson's ratio. In this example, the case where $v = 1/3$, $\rho = 2.0\times10^3$ (kg/m^3), $V_s = 100$ (m/s) and $r_o = 5.64$ (m) is dealt with. The constants are then given by $b_1 = 0.5$, $b_2 = 0.8$ and $b_3 = 0.0$. In this case, the horizontal stiffness $k_H(\Omega)$ turns out to be a constant and the rotational stiffness $k_R(\Omega)$ a function of Ω. Figure 7 shows the plots of the values in the bracket in eqn (35) with respect to the

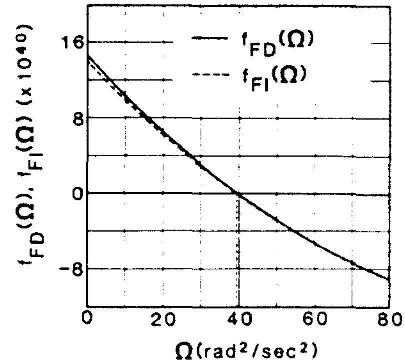

Fig. 8. Plots of $f_{FD}(\Omega)$ and $f_{FI}(\Omega)$ with respect to Ω for shear buildings with frequency-dependent and frequency-independent supports.

dimensionless frequency $a_0 = \omega r_0 / V_s$ where $\omega = \sqrt{\Omega}$. It is therefore said that the stiffnesses $k_H(\Omega)$ and $k_R(\Omega)$ are single-valued non-increasing positive functions of Ω.

A ten-story shear building is considered here. The lumped masses and the moments of inertia of floor masses are prescribed as $m_0 = 90.0 \times 10^3$ (kg), $m_i = 30.0 \times 10^3$ (kg) ($i = 1, \ldots, 10$) and $I_{R0} = 7.5 \times 10^5$ (kg·m^2), $I_{Ri} = 2.5 \times 10^5$ (kg·m^2) ($i = 1, \ldots, 10$). The story heights are prescribed as $h_i = 350.0$ (cm) ($i = 1, \ldots, 10$) and the radii of gyration of columns as a constant. If the fundamental eigenvalue is specified as $\Omega_a = 39.5$ (rad^2/s^2) which corresponds to 1.0 (s) of the fundamental natural period, the corresponding dimensionless frequency a_0 and the dimensionless stiffness of $k_R(\Omega)$ are indicated by the mark □ in Fig. 7. Let $\bar{k}_H = k_H(\Omega_a)$ and $\bar{k}_R = k_R(\Omega_a)$ denote the frequency-independent stiffnesses. Then the optimal solution of the shear building which is supported by the frequency-independent springs $\{\bar{k}_H, \bar{k}_R\}$ and has the fundamental eigenvalue Ω_a is obtained from the design formula by Nakamura and Takewaki (1985) and is given by $\{A_i g_i\} = \{113.8, 111.5, 107.3, 100.9, 92.6, 82.2, 69.8, 55.4, 39.0, 20.5\}$ (N/m).

The solid line in Fig. 8 shows the plots of the function corresponding to eqn (6) with respect to Ω of the shear building of design $\{A_i g_i\}$ with $\{k_H(\Omega), k_R(\Omega)\}$. On the other hand, the broken line in Fig. 8 shows the plots of the function corresponding to eqn (7) with respect to Ω of the shear building of design $\{A_i g_i\}$ with $\{\bar{k}_H, \bar{k}_R\}$. Figure 8 indicates that both of the shear buildings of the same design $\{A_i g_i\}$ with $\{k_H(\Omega), k_R(\Omega)\}$ and $\{\bar{k}_H, \bar{k}_R\}$ have Ω_a as the fundamental eigenvalue. This fact clearly demonstrates the validity of Theorems 1 and 2.

7. CONCLUSIONS

Two theorems have been introduced and proved. In the first theorem it has been proved that an elastic frame supported by members with frequency-dependent stiffnesses has the same set of a fundamental natural frequency and a fundamental eigenvector as that of the same elastic frame supported by members with the corresponding frequency-independent stiffnesses, provided the former stiffnesses are expressed by single-valued non-increasing positive functions of frequency. It has been shown also that the converse theorem holds. These two theorems have established one-to-one correspondence between the design spaces of an ordered set of elastic frames supported by members with frequency-dependent stiffnesses and of the corresponding ordered set of elastic frames supported by those with the corresponding frequency-independent stiffnesses, both with respect to fundamental natural frequency. On the basis of this one-to-one correspondence, it has been concluded that the necessary and sufficient conditions of global optimality for the optimum design problem of the former frames coincide with those of the latter frames.

It has been shown furthermore that, for the ordered set of optimum designs with respect to the prescribed fundamental natural frequency, the first derivative of the objective function is positive throughout that range of frequency for which all the support stiffnesses are expressed as single-valued non-increasing positive functions. Hence it has been concluded that the optimal solution to the optimum design problem for specified fundamental natural frequency is also the optimal solution to the problem subject to the corresponding inequality constraint for frames with those supports. The implication of the theorems has been illustrated through an optimum design of a ten-story plane shear building model supported by two springs with realistic frequency-dependent stiffnesses and by those with frequency-independent stiffnesses.

It should be remarked finally that the two theorems are certainly applicable to any finite element model supported by springs with frequency-dependent stiffnesses.

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